

Budget Feasible Mechanisms on Matroids

Stefano Leonardi¹, Gianpiero Monaco², Piotr Sankowski³, and Qiang Zhang¹

¹ Sapienza University of Rome, Italy, leonardi@dis.uniroma1.it, qzhang@gmail.com

² University of L'Aquila, Italy, gianpiero.monaco@univaq.it

³ University of Warsaw, Poland, sank@mimuw.edu.pl

Abstract. Motivated by many practical applications, in this paper we study *budget feasible mechanisms* where the goal is to procure independent sets from matroids. More specifically, we are given a matroid $\mathcal{M} = (E, \mathcal{I})$ where each ground (indivisible) element is a selfish agent. The cost of each element (i.e., for selling the item or performing a service) is only known to the element itself. There is a buyer with a budget having additive valuations over the set of elements E . The goal is to design an incentive compatible (truthful) budget feasible mechanism which procures an independent set of the matroid under the given budget that yields the largest value possible to the buyer. Our result is a deterministic, polynomial-time, individually rational, truthful and budget feasible mechanism with 4-approximation to the optimal independent set. Then, we extend our mechanism to the setting of matroid intersections in which the goal is to procure common independent sets from multiple matroids. We show that, given a polynomial time deterministic blackbox that returns α -approximation solutions to the matroid intersection problem, there exists a deterministic, polynomial time, individually rational, truthful and budget feasible mechanism with $(3\alpha + 1)$ -approximation to the optimal common independent set.

1 Introduction

Procurement auctions (a.k.a. reverse auctions), often carried out by governments or private companies, deal with the scenarios where a buyer would like to purchase objects from a set of sellers. These objects are not limited to physical items. For instance they can be services provided by sellers. In this work we consider the problem where a buyer with a budget is interested in a set of indivisible objects for which he has additive valuations. We assume that each object is a selfish agent. More specifically, we assume agents have quasi-linear utilities and they are rational (i.e., they aim to maximize the differences between the payments they receive and their true costs). We also restrict ourself to the case where the buyer is constrained to purchase a subset of objects that forms an independent set with respect to an underlying matroid structure. A wide variety of research studies have shown that matroids are linked to many interesting applications, for example, auctions [2,9,13], spectrum market [17], scheduling matroids [8] and house market [14].

One challenge in such procurement auctions involves providing incentives to sellers for declaring their true costs when those costs are their *private information*. A classical mechanism, namely Vickrey-Clark-Groves (VCG) mechanism [7,11,18], provides an intuitive solution to this problem. The VCG mechanism returns a procurement that maximizes the valuation of the buyer and the payments for sellers are their externalities to the procurement. The VCG mechanism is a *truthful* mechanism, i.e., no seller will improve its utility by manipulating its cost regardless the costs declared by others. However, the VCG mechanism also has its drawbacks. One of the drawbacks, which makes VCG mechanism impractical, is that the payments to sellers could be very high. To overcome this problem two different approaches have been proposed and investigated. The first one is studying the *frugality* of mechanisms [12], which studies the minimum payment the buyer needs to pay for a set of objects when sellers are rational utility maximizers. The other approach is developing *budget feasible mechanisms* [16], where the goal is to maximize the buyer's value for the procurement under a given budget when sellers are rational utility maximizers. Singer [16] showed

that budget feasible mechanisms could approximate the optimal procurement that “magically” knows the costs of sellers, when the buyer’s valuation is nondecreasing submodular.

Our Results. The goal of this study is to design budget feasible mechanisms for procuring objects that form an independent set in a given matroid structure. To the best of our knowledge it is the first time that matroid constraints are considered in the budget feasible mechanisms setting examined here. Previous work was mainly devoted to different types of valuations for the buyer (see the Related Work subsection). Our results are positive. In Section 3 we give a deterministic, polynomial time, individually rational, truthful and budget feasible mechanism with 4-approximation to the optimal independent set (i.e., the independent set with maximum value for the buyer under the given budget) within the budget of the buyer when the buyer has additive valuations. To generalize this result we also provide a similar mechanism to procure the intersection of independent sets in multiple matroids. In particular, given a deterministic polynomial time α -approximation algorithm for the matroid intersection problems as a blackbox, in Section 4 we present a deterministic, polynomial time, individually rational, truthful and budget feasible mechanism with $(3\alpha + 1)$ -approximation to the optimal independent set within the budget of the buyer when the buyer has additive valuations. It is also good to know the limitations (e.g. lower bounds) of such budget feasible mechanisms. In particular the lower bound to any deterministic mechanism of $1 + \sqrt{2}$ for additive valuations with one buyer presented in [6] (it is worth noticing that such lower bound do not rely on any computational or complexity assumption), suggests that our mechanisms are not far away from the optimal ones. Finally, budget feasible mechanisms also received a lot of attention when the valuation functions are submodular [16] and XOS [3]. In Section 6 we slightly improve the analysis of the mechanism proposed in [3]. Specifically, we improve the approximation ratio of the mechanism from 768 to 436 by tuning the parameters in the mechanism.

Related Work. The study of budget feasible mechanisms was initiated in [16]. It essentially focuses on the procurement auctions when sellers have private costs for their objects and a buyer aims to maximize his valuation function on subsets of objects, conditioned on that the sum of the payments given to sellers *cannot* exceed a given budget of the buyer. In particular Singer [16] considered budget feasible mechanisms when the valuation function of the buyer is nondecreasing submodular. For general nondecreasing submodular functions, Singer [16] gave a lower bound of 2 for deterministic budget feasible mechanisms and a randomized budget feasible mechanism with 112-approximation. When the valuation function of the buyer is additive, a special class of nondecreasing submodular functions, Singer [16] gave a polynomial deterministic budget feasible mechanism with 6-approximation and a lower bound of 2 for any deterministic budget feasible mechanism. All results were improved in [6], for example, a deterministic budget feasible mechanism with $2 + \sqrt{2}$ -approximation and an improved lower bound of $1 + \sqrt{2}$ for any deterministic budget feasible mechanism for additive valuations were given. Furthermore, Bei et al. [3] gave a 768-approximation mechanism for XOS valuations and extended their study to Bayesian settings. Chan and Chen [5] studied budget feasible mechanisms in the settings in which each seller processes multiple copies of the objects. They gave logarithmic mechanisms for concave additive valuations and sub-additive valuations.

Budget feasible mechanisms are attractive to many communities due to their various applications. In crowdsourcing the goal is to assign skilled workers to tasks when workers have private costs. By injecting some characteristics in crowdsourcing, budget feasible mechanisms have been further developed and improved. For example, Goel et al. [10] developed budget feasible mechanisms that achieve $\frac{2e-1}{e-1}$ -approximation to the optimal social welfare by exploiting the assumption that one worker has limited contribution to the social welfare. Furthermore Anari et al. [1] gave a budget feasible mechanism that achieves a competitive ratio of $1 - 1/e \approx 0.63$ by using the assumption that the cost of any worker is relatively small compared to the budget of the buyer.

Another work close to ours is [4], which studies the “dual” problem of maximizing the revenue by selling the maximum independent set of a matroid. They proposed a truthful ascending auction in which a seller is constrained to sell objects that forms a basis in a matroid.

2 Preliminaries

Matroids. A matroid \mathcal{M} is a pair of (E, \mathcal{I}) where E is a ground set of finite elements and $\mathcal{I} \subseteq 2^E$ consists of subsets of the ground set satisfying the following properties:

- Hereditary property: If $I \in \mathcal{I}$, then $J \in \mathcal{I}$ for every $J \subset I$.
- Exchange property: For any pair of sets $I, J \in \mathcal{I}$, if $|I| < |J|$, then there exists an element $e \in J$ such that $I \cup \{e\} \in \mathcal{I}$.

The sets in \mathcal{I} are called *independent sets*. Given a matroid $\mathcal{M} = (E, \mathcal{I})$ and $T \subseteq E$ is a subset of E , the *restriction* of \mathcal{M} to T , denoted by $\mathcal{M}|T$, is the matroid in which the ground set is T and the independent sets are the independent sets of \mathcal{M} that are contained in T . That is, $\mathcal{M}|T = (T, \mathcal{I}(\mathcal{M}|T))$ where $\mathcal{I}(\mathcal{M}|T) = \{I \subseteq T : I \in \mathcal{I}\}$. Similarly, the *deletion* of \mathcal{M} , denoted by $\mathcal{M} \setminus T$, is the matroid in which the ground set is $E - T$ and the independent sets are the independent sets of \mathcal{M} that do not contain any element in T . That is, $\mathcal{M} \setminus T = (E - T, \mathcal{I}(\mathcal{M} \setminus T))$ where $\mathcal{I}(\mathcal{M} \setminus T) = \{I \subseteq E - T : I \in \mathcal{I}\}$.

Matroid Budget Feasible Mechanisms. In an instance of the matroid budget feasible mechanism design problem, we are given a matroid $\mathcal{M} = (E, \mathcal{I})$ consisting of n ground elements, each of whom is associated with a weight $w_e \in \mathbb{R}_+$. Each element $e \in E$ is also associated with a private cost $c_e \in \mathbb{R}_+$, which is only known to the element itself. Our goal is to design a truthful mechanism that gives incentives to elements for declaring their private costs truthfully and then selects an independent set conditioned on that the total payment given to the elements does not exceed a given budget b . Given an independent set $I \in \mathcal{I}$, the value of the independent set is defined by $w(I) = \sum_{e \in I} w(e)$. We compare the value of the independent set selected by the mechanism against the value of the maximum-value independent set in which the total true cost of elements does not exceed the budget.

We use $\mathbf{w} = \langle w_1, \dots, w_n \rangle$ to denote the weight of the ground elements and use $\mathbf{d} = \langle d_1, \dots, d_n \rangle$ to denote the costs declared by the ground elements. Let τ be the maximum-weight element (breaking ties arbitrarily), that is, $w_\tau = \max_{e \in E} w_e$. We assume that $d_e \in \mathbb{R}_+$ and $d_e \leq b$ for any $e \in E$ since elements with costs greater than b cannot be selected by any mechanism due to the budget constraint. This also implies that no element could improve its utility by declaring $d_i > b$. Given a subset of element T , we use \mathbf{w}_{-T} and \mathbf{d}_{-T} to denote the weight and cost vector excluding elements in T . Similarly, we use \mathbf{w}_T and \mathbf{d}_T to denote the weight and cost vector only including elements in T . For each element $e \in E$, $\text{bb}(e) = \frac{d_e}{w_e}$ is called the *buck-per-bang* rate for element e .⁴

A deterministic mechanism $M = (f, p)$ consists of an allocation function $f : \mathcal{M}, \mathbf{w}, \mathbf{d}, b \rightarrow I \in \mathcal{I}$ and a payment function $p : \mathcal{M}, \mathbf{w}, \mathbf{d}, b \rightarrow \mathbb{R}_+^n$. Given the weights and declared costs of the ground elements, the allocation function returns an independent set in the matroid and the payment function indicates the payments for all elements. Let $\mathbf{f}^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b)$ and $\mathbf{p}^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b)$ be the independent set and payments returned by M , respectively. If element e is in the independent set obtained by M , then $f_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) = 1$. Otherwise, $f_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) = 0$. It is assumed that $p_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) = 0$ if $f_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) = 0$. The utility of an element is the difference between the payment received from the mechanism and its true cost. More specifically, the utility of element e is given by $u_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) = p_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) - f_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) \cdot c_e$.

⁴ $\frac{w_e}{c_e}$ is usually known as the *bang-per-buck* rate. To simplify the presentation, we call $\frac{d_e}{w_e}$ the *buck-per-bang* rate.

Individual Rationality: A mechanism M is *individually rational* if $p_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) - f_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) \cdot d_e \geq 0$ for any \mathcal{M} , any $\mathbf{w} \in \mathbb{R}_+^n$, any $\mathbf{d} \in \mathbb{R}_+^n$, any $b \in \mathbb{R}_+$ and any element $e \in E$. That is, no element in the selected independent set is paid less than the cost it declared.

Truthfulness: A mechanism M is *truthful* if it holds $u_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}_{-e}, c_e, b) \geq u_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}_{-e}, d_e, b)$ for any \mathcal{M} , any $\mathbf{w} \in \mathbb{R}_+^n$, any $\mathbf{d}_{-e} \in \mathbb{R}_+^{n-1}$, any $d_e \in \mathbb{R}_+$, any $c_e \in \mathbb{R}_+$, $b \in \mathbb{R}_+$ and any $e \in E$, where $\mathbf{d}_{-e} = \langle d_1, \dots, d_{e-1}, d_{e+1}, \dots, d_n \rangle$. When the context is clear, we sometimes abuse some notations. For example, here we write $u_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}_{-e}, c_e, b)$ instead of $u_e^M(\mathcal{M}, \mathbf{w}, \langle \mathbf{d}_{-e}, c_e \rangle, b)$. A truthful mechanism prevents any element improving its utility by mis-declaring its cost regardless the costs declared by other elements.

Budget Feasibility: A mechanism M is *budget feasible* if it holds that $\sum_{e \in E} p_e^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b) \leq b$ for any \mathcal{M} , $\mathbf{w} \in \mathbb{R}_+^n$, any $\mathbf{d} \in \mathbb{R}_+^n$ and any $b \in \mathbb{R}_+$.

Competitiveness: A mechanism M is α -*competitive* if $w(f^M(\mathcal{M}, \mathbf{w}, \mathbf{d}, b)) \geq \frac{1}{\alpha} w(\text{OPT}(\mathcal{M}, \mathbf{w}, \mathbf{d}, b))$ for any $\mathbf{w} \in \mathbb{R}_+^n$, $\mathbf{d} \in \mathbb{R}_+^n$ and $b \in \mathbb{R}_+$, where $\text{OPT}(\mathcal{M}, \mathbf{w}, \mathbf{d}, b)$ is the maximum-value independent set in which the total cost of the elements is at most b . We often call $\text{OPT}(\mathcal{M}, \mathbf{w}, \mathbf{d}, b)$ the optimal independent set and simplify it as $\text{OPT}(\mathcal{M}, b)$ throughout the paper when the weights and the costs of elements are clear. Similarly we use $\text{MAX}(\mathcal{M}, \mathbf{w})$, shorten by $\text{MAX}(\mathcal{M})$, to denote the maximum-value independent set in \mathcal{M} without considering the budget constraint.

Simplifying notations. From now on to avoid heavy notations we sometimes simplify the notations. For example we will write f^M, f_e^M, p^M, p_e^M when the inputs of the mechanism are clear. And we will use $\text{OPT}(\mathcal{M} \setminus T, b)$ instead of $\text{OPT}(\mathcal{M} \setminus T, \mathbf{w}_{-T}, \mathbf{d}_{-T}, b)$ to denote the optimal independent set in matroid $\mathcal{M} \setminus T$. Similarly we will use $\text{OPT}(\mathcal{M}|T, b)$ instead of $\text{OPT}(\mathcal{M}|T, \mathbf{w}_T, \mathbf{d}_T, b)$ to denote the optimal independent set in matroid $\mathcal{M}|T$. Furthermore we use $\text{MAX}(\mathcal{M} \setminus T)$ instead of $\text{MAX}(\mathcal{M} \setminus T, \mathbf{w}_{-T})$ to denote the maximum-value independent set in $\mathcal{M} \setminus T$ without considering the costs of the elements and the budget.

3 Mechanisms for Matroids

In this section we provide our main result. We give a deterministic, polynomial time, individually rational, truthful and budget feasible mechanism that is 4-approximating the optimal independent set. Before providing the mechanism we discuss some intuition that guides us in the design of Mechanism 1. First imagine that there exists an element with a very high weight, i.e., any independent set without this element results in a poor value compared to the optimal independent set. In this case that element may strategically declare a high cost in order to increase its utility as it knows that any competitive mechanism has to select it. To avoid that this happens we remove element τ (i.e., the element with the largest weight) from the matroid via matroid deletion operation, and compare it with the independent set computed later by the mechanism. Second we observe that most of the existing budget feasible mechanisms adopt proportional payment schemes, where elements (i.e., agents) are paid proportionally according to their contribution in the solution. In other words in a proportional payment scheme there is an uniform price such that the payments for elements in the solution are the products of their contribution and this price. In addition greedy algorithms are commonly used in matroid systems. Combining these two observations our plan is to start from a high price and compute the maximum-value independent set in the matroid at each iteration. If there is enough budget to pay this independent set at the current price then we proceed to the final step of the mechanism. Otherwise we reduce the price and remove an element from the

matroid. The buck-per-bang rate of that element becomes an upper bound of the payment on each contribution in the next iteration. The mechanism performs the procedure described above until the payment of the maximum-value independent set is within budget b . As we will show next, if the value of the optimal independent set does not come from a single element, we are able to retain most of the value of the optimal independent set after removing those elements. Finally, we show that returning the better solution between the maximum-value independent set found and element τ approximates the value of the optimal independent set within a factor of 4.

Mechanism 1: A budget feasible mechanism for procuring independent sets in matroids

Input: $\mathcal{M} = (E, \mathcal{I})$, $\mathbf{w}, \mathbf{d}, b$

Output: \mathbf{f}, \mathbf{p}

- 1 Sort elements in $E - \tau$ in a non-increasing order of buck per bang, i.e. $\text{bb}(i) \geq \text{bb}(j)$ if $i < j$, break ties arbitrarily;
 - 2 Let $\text{bb}(0) = +\infty$, $i = 1$ and $T = \emptyset$;
 - 3 Set $r = \text{bb}(i)$;
 - 4 **while** $w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau))) \cdot r > b$ **do**
 - 5 $T = T \cup \{i\}$ and $i = i + 1$;
 - 6 $r = \min\{\frac{b}{w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau)))}, \text{bb}(i - 1)\}$;
 - 7 **if** $w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau))) > w_\tau$ **then**
 - 8 For each $e \in E$, if $e \in \text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, $f_e = 1$ and $p_e = r \cdot w_e$. Otherwise, $f_e = 0$ and $p_e = 0$;
 - 9 **else**
 - 10 $f_\tau = 1, p_\tau = b$. For edge $e \in E - \tau$, $f_e = 0, p_e = 0$;
 - 11 **return** \mathbf{f}, \mathbf{p} ;
-

Theorem 3.1. *Mechanism 1 is a deterministic, polynomial time, individually rational, truthful and budget feasible mechanism that is 4-competitive against the optimal independent set given a budget.*

3.1 Approximation

Recall that T is the set of elements removed from the matroid. $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ is the independent set found when Mechanism 1 stops, and it is also the maximal-value independent set in matroid $\mathcal{M} \setminus (T \cup \tau)$. The roadmap of the proof is to first show that, the independent set $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ well approximates the optimal independent set in matroid $\mathcal{M} \setminus \tau$. Next we show that returning the maximum between τ and $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ gives 4-approximation to the optimal independent set in matroid \mathcal{M} .

Lemma 3.1. *Given any $\mathcal{M}, \mathbf{w}, \mathbf{d}, b$, when Mechanism 1 stops, it holds*

$$w(\text{OPT}(\mathcal{M} \setminus \tau, b)) \leq 2w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau))) + w_\tau$$

Proof. It is trivial to see that this lemma holds when τ is the only element in matroid \mathcal{M} . The rest of the proof uses a similar idea in [10] and is divided into two cases depending on whether the full budget b is spent or not. Consider $E - \{\tau\}$ is partitioned into two disjoint sets, $E - \{\tau\} - T$ and T . The value of maximum-value independent set $w(\text{OPT}(\mathcal{M} \setminus \tau, b))$ is bounded by

$$w(\text{OPT}(\mathcal{M}|T, b)) + w(\text{OPT}(\mathcal{M} \setminus (T \cup \tau), b))$$

As the buck-per-bang is at least r for every element in T , the optimal independent set given a budget b in $\mathcal{M}|T$, i.e. $w(\text{OPT}(\mathcal{M}|T, b))$, is at most b/r . When the full budget is spent, the independent set f^M is b/r in Mechanism 1. On the other hand, f^M is the maximum-value independent set in $\mathcal{M} \setminus (T \cup \tau)$. It implies that $w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau))) \geq w(\text{OPT}(\mathcal{M} \setminus (T \cup \tau), b))$. The above analysis concludes that

$$w(\text{OPT}(\mathcal{M} \setminus \tau, b)) \leq 2w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau)))$$

Now we turn to the case that some budget is left in Mechanism 1. Note that it happens because $r = \text{bb}(i - 1)$ (see Line 6) during the execution of Mechanism 1. Since Mechanism 1 does not stop when $r = \text{bb}(i - 1)$, it implies that the maximum-value independent set found was not budget feasible at previous iteration. After removing element $i - 1$, the maximum-value independent set becomes budget feasible when $r = \text{bb}(i - 1)$. These together imply

$$w(\text{MAX}(\mathcal{M} \setminus (T' \cup \tau))) \cdot \text{bb}(i - 1) > b > w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau))) \cdot \text{bb}(i - 1)$$

where $T' = T - \{i - 1\}$. This further implies that budget left is at most $\text{bb}(i - 1) \cdot w_{i-1}$. By the similar argument as in previous case, the optimal independent set in $\mathcal{M}|T$ is at most b/r , while the value of the independent set $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ is at least $(b - \text{bb}(i - 1) \cdot w_{i-1})/r$, which is at least $b/r - w_{i-1}$. Therefore, we have

$$w(\text{OPT}(\mathcal{M} \setminus \tau, b)) \leq 2w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau))) + w_{i-1}$$

Substituting w_{i-1} with w_τ completes the proof. \square

Next, we show that returning the maximum between τ and $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ is 4-competitive against the optimal independent set in \mathcal{M} .

Lemma 3.2. *Given any $\mathcal{M}, \mathbf{w}, \mathbf{d}, b$, the independent set returned by Mechanism 1, i.e., the maximum between τ or $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, is 4-competitive against the optimal independent set.*

Proof. The optimal independent set in \mathcal{M} is bounded by

$$w(\text{OPT}(\mathcal{M}, b)) \leq w_\tau + w(\text{OPT}(\mathcal{M} \setminus \tau, b))$$

By Lemma 3.1, we have

$$w(\text{OPT}(\mathcal{M}, b)) \leq 2w_\tau + 2w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau)))$$

Therefore, the maximum between τ and $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ approximates the optimal independent set within a factor of 4. \square

3.2 Truthfulness

In this section, we will show that Mechanism 1 is truthful. It is easy to see that element τ cannot benefit by manipulating its cost.

Lemma 3.3. *The element with the maximum weight, i.e., element τ , could not improve his utility by declaring cost $d_\tau \neq c_\tau$.*

Proof. If Mechanism 1 returns element τ when τ declares his true cost, then τ gets a payment of b so that there is no incentive for him to declare other cost. On the other hand, when Mechanism 1 returns $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, declaring a different cost will not change the outcome as it is still the element with the largest weight and $w_\tau < w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau)))$. \square

Next we show that no edge in $E - \tau$ could improve his utility by mis-declaring its cost. The proof relies on the analysis of different cases. The first case shows that those removed element in Mechanism 1 cannot benefit by manipulating their costs.

Lemma 3.4. *Assume an element k in T when it declares its cost truthfully. Then, element k could not improve his utility by declaring a cost $d_k \neq c_k$.*

Proof. As $k \in T$, we know that element k is not in the independent set returned by Mechanism 1. Hence, his utility is zero. It implies, when element k is considered, that is, $r = \text{bb}(k)$, it holds that $w(\text{MAX}(\mathcal{M} \setminus (T^k \cup \tau))) \cdot r > b$ where T^k denotes the set of elements removed until k is considered. Consider that element k declares a higher cost $d_k > c_k$ and it is considered earlier at the h^{th} iteration where $h \leq k$. Equivalently speaking, k becomes the element with the h^{th} largest buck-per-bang rate. In this case, Mechanism 1 will not stop until k is considered. Note that the declared costs are not involved in computing maximum-value independent sets. The maximum-value independent sets computed are exactly the same as declaring truthfully until the h^{th} iteration. Moreover, the maximum-value independent set is also same in the h^{th} iteration since the remaining elements in the matroid are the same. As r in the h^{th} is equal to or greater than before, the maximum-value independent set is not budget feasible. It implies that element k will be removed from the matroid and it will never be included in an independent set in Mechanism 1. Therefore, its utility is zero.

We use similar arguments to show that element k can not improve his utility by declaring a smaller cost. Consider that element k declares a smaller cost $d_k < c_k$ and it is considered at the h^{th} iteration where $h \geq k$. Let us focus on how Mechanism 1 performs. Until the k^{th} iteration, maximum-value independent sets are the same as k declaring its cost truthfully and they are not budget feasible. Next, the maximum-value independent set in k^{th} is also the same as before. If the independent set is budget feasible, then we know that $\frac{b}{w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau)))}$ is strictly less than $\text{bb}(k) = \frac{w_k}{c_k}$. It is because that the mechanism does not terminate at $r = \frac{w_k}{c_k}$ when element k declares truthfully. Therefore, even element k is in this independent set, the payment will be strictly less than his true cost. On the other hand, if the independent set is not budget feasible, then the mechanism will update its upper bound of payment for each contribution in the next iteration. The new upper bound is at most $\text{bb}(k) = \frac{w_k}{c_k}$. It implies element k will never get a payment greater than his true cost. \square

In the second case we show that the remaining elements that are not included in the independent set cannot benefit by manipulating their costs.

Lemma 3.5. *Assume an element k is in $E - \tau - T - \text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ when it declares its cost truthfully. Then, element k could not improve his utility by declaring a cost $d_k \neq c_k$.*

Proof. As $k \notin \text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, we know that the utility of element k is zero. Suppose that the mechanism terminates at the h^{th} round when element k declares its cost truthfully. Consider that element k declares a higher cost $d_k > c_k$ and it is consider at the l^{th} iteration where $l < h$. In this case, Mechanism 1 will not stop before or at the l^{th} iteration since the maximum-value independent sets computed are exactly the same as k declaring its cost truthfully and they are not budget feasible. It implies that element k will be removed from the matroid and it will never be included in an independent set in Mechanism 1 by declaring a larger cost. Therefore, its utility is still zero.

Secondly, consider that element k declares a cost $d_k \neq c_k$ and it is considered at the h^{th} iteration. In this case, Mechanism 1 will not stop before the h^{th} iteration because the maximum-value independent sets are not feasible. The maximum-value independent in the h^{th} iteration is

the same as k declaring its cost truthfully since the remaining elements are the same. Therefore, if the independent set is budget feasible, the mechanism will compute the same independent set and payments. Otherwise, element k will be removed as it must be the element with the h^{th} largest buck-per-bang rate. Element k cannot benefit in any case.

Finally, consider that element k declares a cost $d_k \neq c_k$ and it is considered after the h^{th} iteration. In this case, Mechanism 1 will compute the same independent set and payments. \square

By similar arguments, we show that elements in the independent set cannot benefit by manipulating their costs.

Lemma 3.6. *Assume an element k is in $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$ when it declares its cost truthfully. Then, element k could not improve his utility by declaring a cost $d_k \neq c_k$.*

Proof. The proof is exactly the same as Lemma 3.5. Suppose that the mechanism terminates at the h^{th} round when element k declares its cost truthfully. As $k \in \text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, we know the payment of element k is $r \cdot w_k$. Hence, his utility is $r \cdot w_k - c_k$. Consider that element k declares a higher cost $d_k > c_k$ and it is considered at the l^{th} iteration where $l < h$. Similar to Lemma 3.5, in this case Mechanism 1 will not stop before or at the l^{th} iteration since the maximum-value independent sets computed are exactly the same and they are not budget feasible. It implies that element k will be removed from the matroid and it will never be included in an independent set in Mechanism 1. Therefore, its utility becomes zero.

Secondly, consider that element k declares a cost $d_k \neq c_k$ and it is considered at the h^{th} iteration. In this case, Mechanism 1 will not stop before the h^{th} iteration because the maximum-value independent sets are not feasible. The maximum-value independent in the h^{th} round is the same as k declaring its cost truthfully since the remaining elements are the same. Therefore, if the independent set is budget feasible, the mechanism will compute the same independent set and payments. Otherwise, element k will be removed. Element k cannot benefit in any case.

Finally, consider that element k declares a cost $d_k \neq c_k$ and it is considered after the h^{th} iteration. In this case, Mechanism 1 will compute the same independent set and payments. \square

3.3 Individual Rationality

When Mechanism 1 returns τ , the utility of τ is non-negative as c_τ is at most b . The utilities for other edges are zero. When Mechanism 1 returns $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, for any element $e \in \text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, that is, $f_e = 1$, its utility is $r \cdot w_e - c_e$ which is non-negative since $r \geq \text{bb}(e)$. For other edges, their utilities are zero.

3.4 Budget Feasibility

When Mechanism 1 returns τ , it only pays b to edge τ . Hence, it is budget feasible. On the other hand, when Mechanism 1 returns $\text{MAX}(\mathcal{M} \setminus (T \cup \tau))$, r is used as payment per contribution. As $r = \min\{\frac{b}{w(\text{MAX}(\mathcal{M} \setminus (T \cup \tau)))}, \text{bb}(i-1)\}$, it guarantees the budget feasibility.

3.5 Remarks

In Mechanism 1, we iteratively compute the maximum-value independent set (e.g. Line 4). In the case that the maximum-value independent set is not unique, we assume there is a deterministic tie-breaking rule. Note that all the results still hold under this assumption. For example, the truthfulness of the mechanism will not be compromised since the maximum-value independent set only consider the weights of the elements that is the public knowledge.

4 Mechanisms for matroid intersections

In this section we extend our mechanism to matroid intersections. The matroid intersection problem (i.e., finding the maximum-value common independent set) is NP-hard in general when more than three matroids are involved. Some interesting cases of matroid intersection problems can be solved efficiently (i.e., they can be formulated as the intersection of two matroids), for example, matchings in bipartite graphs, arborescences in directed graphs, spanning forests in undirected graphs, etc. Nevertheless we point out that a very similar mechanism to the one presented in last section achieves a 4 approximation for the case when, instead of a matroid, we are given an undirected weighted (general) graph where the selfish agents are the edges of the graph and the buyer wants to procure a matching under the given budget that yields the largest value possible to him.

For general matroid intersections, our main result is the following. Given a deterministic polynomial time blackbox APX that achieves an α -approximation to k -matroid intersection problems, we provide a polynomial time, individually rational, truthful and budget feasible deterministic mechanism that is $(3\alpha + 1)$ -competitive against the maximum-value common independent set. The mechanism is similar to Mechanism 1 by changing MAX to APX. It is well-known that the VCG payment rule does not preserve the property of truthfulness in the presence of approximated solutions (i.e., non-optimal outcome). However unlike the VCG mechanism, we show that Mechanism 2 preserves its truthfulness when APX is used. We believe that this result will make our contribution more practical.

Mechanism 2: A budget feasible mechanism for procuring independent sets in matroid intersections

Input: $\mathcal{M} = (E, \mathcal{I}), \mathbf{w}, \mathbf{d}, b$
Output: \mathbf{f}, \mathbf{p}

- 1 Sort elements in $E - \tau$ in a non-increasing order of buck per bang, i.e. $\text{bb}(i) \geq \text{bb}(j)$ if $i < j$, break ties arbitrarily;
- 2 Let $\text{bb}(0) = +\infty$, $i = 1$ and $T = \emptyset$;
- 3 Set $r = \text{bb}(i)$;
- 4 **while** $w(\text{APX}(\mathcal{M} \setminus T)) \cdot r > b$ **do**
- 5 $T = T \cup \{i\}$ and $i = i + 1$;
- 6 $r = \min\{\frac{b}{w(\text{APX}(\mathcal{M} \setminus T))}, \text{bb}(i - 1)\}$;
- 7 **if** $w(\text{APX}(\mathcal{M} \setminus T)) > w_\tau$ **then**
- 8 For each $e \in E$, if $e \in \text{APX}(\mathcal{M} \setminus T)$, $f_e = 1$ and $p_e = r \cdot w_k$. Otherwise, $f_e = 0$ and $p_e = 0$;
- 9 **else**
- 10 $f_\tau = 1, p_\tau = b$. For edge $e \in E - \tau$, $f_e = 0, p_e = 0$;
- 11 **return** \mathbf{f}, \mathbf{p} ;

4.1 Matroid intersections

Given k -matroid $\mathcal{M}_1, \dots, \mathcal{M}_k$, let $\mathcal{M} = (E, \mathcal{I})$ be the “true matroid” where E is the common ground elements and $\mathcal{I} = \bigcap_j \mathcal{I}_j$ is the “true independent sets”. Similar as the notations we used before, let $\text{OPT}(\mathcal{M} \setminus \mathcal{T}, b)$ and $\text{OPT}(\mathcal{M} | \mathcal{T}, b)$ denote the optimal independent set satisfying the budget constraint in matroid $\mathcal{M} \setminus \mathcal{T}$ and $\mathcal{M} | \mathcal{T}$, respectively. Let $\text{APX}(\mathcal{M} \setminus \mathcal{T}, b)$ be the maximum-value independent set in matroid $\mathcal{M} \setminus \mathcal{T}$ returned by the α -approximation algorithm.

4.2 Obtaining $O(\alpha)$ approximation

We show the following key lemma, which is similar to Lemma 3.1 and implies the approximation of our mechanism for matroid intersections.

Lemma 4.1. *Given any $\mathcal{M}, \mathbf{w}, \mathbf{d}, b$, when Mechanism 2 stops, it holds*

$$w(\text{OPT}(\mathcal{M} \setminus \tau, b)) \leq 2 \cdot \alpha \cdot w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) + \alpha \cdot w_\tau$$

Proof. The proof has the same spirit as the proof of Lemma 3.1. We consider two cases depending on whether the full budget b is spent or not. Consider $E - \{\tau\}$ is partitioned into two disjoint sets, $E - \{\tau\} - T$ and T . Similar to Lemma 3.1, when the full budget is spent, we get

$$\begin{aligned} w(\text{OPT}(\mathcal{M} \setminus \tau, b)) &\leq w(\text{OPT}(\mathcal{M}|T, b)) + w(\text{OPT}(\mathcal{M} \setminus (T \cup \tau), b)) \\ &\leq \frac{b}{r} + \alpha \cdot w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) \\ &\leq (\alpha + 1) \cdot w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) \end{aligned}$$

When there is some budget left in Mechanism 2, the analysis involves one more step compared to Lemma 3.1 although the idea is still to bound the budget left. Since Mechanism 2 does not stop when $r = \text{bb}(i - 1)$, it implies that the independent set returned by APX was not budget feasible at previous iteration. It further implies that the maximum-value independent set is not budget feasible either if the payment per weight is r . After removing element $i - 1$, the independent set returned by APX becomes budget feasible when $r = \text{bb}(e_{i-1})$. These together imply

$$w(\text{MAX}(\mathcal{M} \setminus (T' \cup \tau))) \cdot \text{bb}(i - 1) \geq w(\text{APX}(\mathcal{M} \setminus (T' \cup \tau))) \cdot \text{bb}(i - 1) > b > w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) \cdot \text{bb}(i - 1)$$

where $T' = T - \{i - 1\}$. As the sum of $w(\text{APX}(\mathcal{M} \setminus (T \cup \tau)))$ and $w(i - 1)$ is at least $\frac{1}{\alpha}$ fraction of $w(\text{MAX}(\mathcal{M} \setminus (T' \cup \tau)))$, we get

$$(w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) + w_{i-1}) \cdot \text{bb}(i - 1) \geq \frac{1}{\alpha} \cdot w(\text{MAX}(\mathcal{M} \setminus (T' \cup \tau))) \cdot \text{bb}(i - 1) > \frac{b}{\alpha}$$

Hence, we get $w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) + w_{i-1} > \frac{b}{\alpha \cdot \text{bb}(i-1)}$. Finally,

$$\begin{aligned} \text{OPT}(\mathcal{M} \setminus \tau, b) &\leq w(\text{OPT}(\mathcal{M}|T, b)) + w(\text{OPT}(\mathcal{M} \setminus (T \cup \tau), b)) \\ &\leq \frac{b}{\text{bb}(i - 1)} + \alpha \cdot w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) \\ &\leq 2 \cdot \alpha \cdot w(\text{APX}(\mathcal{M} \setminus (T \cup \tau))) + \alpha \cdot w_{i-1} \end{aligned}$$

Substituting w_{i-1} with w_τ completes the proof. \square

Now, we show the competitive ratio of Mechanism 2

Lemma 4.2. *Given any $\mathcal{M}, \mathbf{w}, \mathbf{d}, b$, the independent set returned by Mechanism 2, i.e., the maximum between τ or $\text{APX}(\mathcal{M} \setminus (T \cup \tau))$, is 4α -competitive against the optimal independent set.*

Proof. The optimal independent set in \mathcal{M} is bounded by

$$w(\text{OPT}(\mathcal{M}, b)) \leq w_\tau + w(\text{OPT}(\mathcal{M} \setminus \tau, b))$$

By Lemma 3.1, we have

$$w(\text{OPT}(\mathcal{M}, b)) \leq (\alpha + 1)w_\tau + 2 \cdot \alpha \cdot w(\text{APX}(\mathcal{M} \setminus (T \cup \tau)))$$

Therefore, the maximum between τ and $\text{APX}(\mathcal{M} \setminus (T \cup \tau))$ approximates the optimal independent set within a factor of $3\alpha + 1$. \square

4.3 Preserving the truthfulness

In this section, we will show that replacing MAX by APX preserve the truthfulness of the mechanism for matroid intersections. The reason behind is that the mechanism works in a greedy fashion and at each iteration the cost declared by elements *does not* affect the independent set computed in the mechanism. The property of the truthfulness relies on the greedy approach instead of the optimality of the independent set. Informally speaking, if an element declares a cost rather than its true cost, its utility will remain the same or it will get removed. The proofs are similar to the proofs in Section 3.2.

Lemma 4.3. *Assume an element $k \in T$ when it declares its cost truthfully. Then, element k could not improve his utility by declaring a cost $d_k \neq c_k$.*

Proof. As $k \in T$, we know that element k is not in the independent set returned by Mechanism 2. Hence, his utility is zero. It implies, when element k is considered, that is, $r = \text{bb}(k)$, we get $w(\text{APX}(\mathcal{M} \setminus (T^k \cup \tau))) \cdot r > b$ where T^k denotes the set of elements removed until k is considered. Consider that element k declares a higher cost $d_k > c_k$ and it is considered earlier at the h^{th} iteration where $h \leq k$. Equivalently speaking, k becomes the element with the h^{th} largest buck-per-bang rate. In this case, Mechanism 2 will not stop until k is considered as the independent sets computed in APX are the same as k declaring its cost truthfully. Moreover, the independent set is also the same in the h^{th} iteration as the remaining elements are the same. As r in the h^{th} is equal to or greater than before, the independent set is not budget feasible. It implies that element k will be removed from the matroid. It concludes that element k will never be included in an independent set in Mechanism 1. Therefore, its utility is still zero.

On the other hand, consider that element k declares a smaller cost $d_k < c_k$ and it is considered at the h^{th} iteration where $h \geq k$. Let us focus on how Mechanism 2 performs. Until the k^{th} iteration, the independent sets computed in APX are the same as k declaring its cost truthfully and they are not budget feasible. Next, the independent set in the k^{th} iteration is the same as before. If the independent set is budget feasible, then we know that r is strictly less than $\text{bb}(k) = \frac{w_k}{c_k}$. It is because that the mechanism does not terminate at $r = \frac{w_k}{c_k}$ when element k declares truthfully. Therefore, even element k is in this independent set, the payment will be strictly less than his true cost. On the other hand, if the independent set is not budget feasible, the mechanism will update its upper bound of payment. The new upper bound is at most $\text{bb}(k) = \frac{w_k}{c_k}$. It implies element k will never get a payment greater than his true cost. \square

Lemma 4.4. *Assume an element k is in $E - \tau - T - \text{APX}(\mathcal{M} \setminus (T \cup \tau))$ when it declares its cost truthfully. Then, element k could not improve his utility by declaring a cost $d_k \neq c_k$.*

Proof. As $k \notin \text{APX}(\mathcal{M} \setminus (T \cup \tau))$, we know that the utility of element k is zero. Suppose that the mechanism terminates at the h^{th} round when element k declares its cost truthfully. Consider that element k declares a higher cost $d_k > c_k$ and it is considered at the l^{th} iteration where $l < h$. In this case, Mechanism 2 will not stop before or at the l^{th} iteration since the independent sets computed in APX are exactly the same as k declaring its cost truthfully and they are not budget feasible. It implies that element k will be removed from the matroid and it will never be included in an independent set in Mechanism 2. Therefore, its utility is still zero.

Secondly, consider that element k declares a cost $d_k \neq c_k$ and it is considered at the h^{th} iteration. In this case, Mechanism 2 will not stop before the h^{th} iteration because the independent sets are not feasible. The independent in the h^{th} iteration is the same as k declaring its cost truthfully since the remaining elements are the same. Therefore, if the independent set is budget feasible, the mechanism will compute the same independent set and payments. Otherwise, element k will

be removed as it must be the element with the h^{th} largest buck-per-bang rate. Element k cannot benefit in any case.

Finally, consider that element k declares a cost $d_k \neq c_k$ and it is considered after the h^{th} iteration. In this case, Mechanism 2 will compute the same independent set and payments. \square

Lemma 4.5. *The element with the maximum weight, i.e., element τ , could not improve his utility by declaring a cost $d_\tau \neq c_\tau$.*

Proof. If Mechanism 2 returns element τ when τ declares his true cost, then τ gets a payment of b so that there is no incentive for him to declare other cost. On the other hand, when Mechanism 2 returns $\text{APX}(\mathcal{M} \setminus (T \cup \tau))$, declaring a different cost will not change the outcome as it is still the element with the largest weight and $w_\tau < w(\text{APX}(\mathcal{M} \setminus (T \cup \tau)))$. \square

Lemma 4.6. *Assume an element k is in $\text{APX}(\mathcal{M} \setminus (T \cup \tau))$ when it declares its cost truthfully. Then, element k could not improve his utility by declaring a cost $d_k \neq c_k$.*

Proof. The proof is exactly the same as Lemma 4.4. Suppose that the mechanism terminates at the h^{th} round when element k declares its cost truthfully. As $k \in \text{APX}(\mathcal{M} \setminus (T \cup \tau))$, we know the payment of element k is $r \cdot w_k$. Hence, his utility is $r \cdot w_k - c_k$. Consider that element k declares a higher cost $d_k > c_k$ and it is considered at the l^{th} iteration where $l < h$. Similar to Lemma 4.4, in this case Mechanism 2 will not stop before or at the l^{th} iteration since the independent sets computed in APX are exactly the same as k declaring its cost truthfully and they are not budget feasible. It implies that element k will be removed from the matroid and it will never be included in an independent set in Mechanism 2. Therefore, its utility becomes zero.

Secondly, consider that element k declares a cost $d_k \neq c_k$ and it is considered at the h^{th} iteration. In this case, Mechanism 2 will not stop before the h^{th} iteration because the maximum-value independent sets are not feasible. The independent in the h^{th} round is the same as k declaring its cost truthfully since the remaining elements are the same. Therefore, if the independent set is budget feasible, the mechanism will compute the same independent set and payments. Otherwise, element k will be removed. Element k cannot benefit in any case.

Finally, consider that element k declares a cost $d_k \neq c_k$ and it is considered after the h^{th} iteration. In this case, Mechanism 2 will compute the same independent set and payments. \square

5 Applications

In this section we briefly discuss some applications of our results.

Uniform Matroid Additive valuation has been studied in the design of budget feasible mechanisms, e.g. [16,6]. In such settings a buyer would like to maximize his valuation by procuring items under the constraint that his payment is at most his budget. Our result generalizes to the case where the buyer has not only the budget constraint but also has a limit on the number of items he can buy. For example hiring people in companies is not only constraint by budgets but also limited by the office space.

Scheduling Matroid Our mechanism could be used to purchase processing time in the context of job scheduling. One special case is the following. Each job is associated with a deadline and a profit, and requires a unit of processing time. As jobs may conflict with each other, only one job can be scheduled at the same time. The buyer would like to maximize his profit by completing jobs under the constraint that he does not spend more than his budget in purchasing processing time.

Spectrum Market Tse and Hanly [17] showed that the set of achievable rates in a Gaussian multiple-access, known as the Cover-Wyner capacity region, forms a polymatroid. It is known there

is a pseudopolynomial reduction from polymatroids to matroids [15]. Therefore, our mechanism can be used to purchase transmission rates by tele-communication companies.

6 XOS functions

Budget feasible mechanisms received a lot of attention also when the valuations functions are submodular [16] and XOS [3]. In this study, we also slightly improve the analysis of the mechanism proposed in [3]. Specifically, we improve the approximation of the mechanism from 768 to 436 by tuning the parameters in the mechanism.

Theorem 6.1. *There exists a randomized universally truthful mechanisms that provides a 436-approximation ratio for XOS valuation functions.*

6.1 Model

We are given a set E consisting of n elements and a budget b . Each element $e \in E$ has a private cost c_e . For any subset $S \subseteq E$, there is publicly known valuation function $v(S)$ that indicates the value of S . In this section, we are interested in XOS valuation functions. More precisely, a function $v(\cdot)$ is XOS if

$$v(S) = \max\{f_1(S), f_2(S), \dots, f_m(S)\} \quad \text{for any } S \subseteq E$$

where each $f_k(\cdot)$ is a nonnegative additive function.

Our goal is to design truthful mechanisms that give elements incentives to declare their true private costs. Meanwhile, mechanisms aim to choose a set of elements within the budget and maximize the value of the chosen agents. Unlike matroids, the mechanism is allowed to select any subset of elements. We compare our mechanisms against the optimal mechanism which always magically knows the private costs of elements. The optimal mechanism returns a set of elements such that the aggregated cost of elements is at most budget b and the value of the elements is maximized. We compare the value of elements chosen by our mechanisms against the value of elements returned by the optimal mechanism.

Similar as truthful mechanisms, when a mechanism is randomized, that is, outputting a distribution over a set of outcomes, we call a randomized mechanism universally truthful if it takes a distribution over deterministic truthful mechanisms.

In the mechanism and its analysis presented in the following sections, we will often consider the optimal solution in a restricted set of elements. Given $S \subseteq E$, let $\text{OPT}(S)$ be the optimal solution when only elements in S are the input of the problem. For example, the optimal solution of the problem is denoted by $\text{OPT}(E)$. We will simply use OPT to denote the optimal solution of the problem, i.e., $\text{OPT} = \text{OPT}(E)$. Let f^* be the additive function in the XOS definition of $v(\cdot)$ with $f^*(\text{OPT}) = v(\text{OPT})$.

6.2 Key Lemmas

Lemma 6.1. *Assume that $f^*(e) \leq \frac{1}{\alpha} f^*(\text{OPT})$ for all $e \in \text{OPT}$, then there exists two disjoint sets $S_1, S_2 \subset E$ such that $v(S_1) \geq \frac{\alpha-1}{2\alpha} f^*(\text{OPT})$ and $v(S_2) \geq \frac{\alpha-1}{2\alpha} f^*(\text{OPT})$.*

Proof. We give a constructive proof for this lemma. In the next paragraph, we will show a way to construct sets S_1 and S_2 such that $f^*(S_1) \geq \frac{\alpha-1}{2\alpha} f^*(\text{OPT})$ and $f^*(S_2) \geq \frac{\alpha-1}{2\alpha} f^*(\text{OPT})$. Given that

$v(S_1) \geq f^*(S_1)$ and $v(S_2) \geq f^*(S_2)$ implied by the definition of XOS functions, the lemma directly follows.

Consider an arbitrary order of elements in OPT, we keep adding elements to S_1 until that $f^*(S_1) \geq \frac{\alpha-1}{2\alpha} f_{OPT}^*(OPT)$. Then, the rest of agents are included in S_2 . By the assume that $f^*(e) \leq \frac{1}{\alpha} f^*(OPT)$ for all $e \in OPT$, it implies that $f^*(S_1) \leq \frac{\alpha+1}{2\alpha} f^*(OPT)$. Finally, since $f^*(S_1) + f^*(S_2) = f^*(OPT)$, we have $f^*(S_2) \geq \frac{\alpha-1}{2\alpha} f^*(OPT)$.

Next, we show that by partitioning elements uniformly random into two groups, we can have a good approximation to the optimal solution at both groups in expectation. The proof shares the same spirit as Lemma 2.1 in [3].

Lemma 6.2. *Assume that $f^*(e) \leq \frac{1}{\alpha} f_{OPT}^*(OPT)$ for all $e \in OPT$. Furthermore, suppose that E is divided uniformly at random into two groups T_1 and T_2 . Then, with probability of at least $\frac{1}{2}$, it holds that $v(T_1) \geq \frac{\alpha-1}{4\alpha} f^*(OPT)$ and $v(T_2) \geq \frac{\alpha-1}{4\alpha} f^*(OPT)$.*

Proof. Let S_1, S_2 be two disjoint sets such that $f^*(S_1) \geq \frac{\alpha-1}{2\alpha} f^*(OPT)$ and $f^*(S_2) \geq \frac{\alpha-1}{2\alpha} f^*(OPT)$. Consider $X_1 = S_1 \cap T_1, Y_1 = S_2 \cap T_1, X_2 = S_1 \cap T_2, Y_2 = S_2 \cap T_2$. As partitioning S_1 into X_1, Y_1 and partitioning S_2 into X_2, Y_2 are independent to each other. Therefore, with probability $\frac{1}{2}$, the most valuable parts of S_1 and S_2 will get into different sets T_1 and T_2 , respectively. Thus the lemma follows.

6.3 Mechanism

XOS-MECHANISM-MAIN(α, β):

1. W.p. $\frac{1}{2}$, pick the most value element and pay him b . W.p. $\frac{1}{2}$, continue.
2. Divide elements independently at random with probability $\frac{1}{2}$ into two set T_1 and T_2 .
3. Compute an optimal solution $OPT(T_1)$ for elements in T_1 given budget b .
4. Set a threshold $t = \frac{v(OPT(T_1))}{\beta \cdot b}$.
5. Find a set $S^* \subseteq T_2$ such that

$$S^* \in \arg \max_{S \subseteq T_2} \{v(S) - t \cdot c(S)\}$$

where $c(S) = \sum_{e \in S} c_e$.

6. Let f be the additive function in the XOS definition of $v(\cdot)$ with $f(S^*) = v(S^*)$.
7. Run ADDITIVE-MECHANISM for f with respect to set S^* and budget b .
8. Output the result of ADDITIVE-MECHANISM.

Lemma 6.3 (Claim 3.1 in [3]). *For any $S \subseteq S^*, f(S) - t \cdot c(S) \geq 0$.*

Lemma 6.4. *XOS-MECHANISM-MAIN has a 436 -approximation ratio.*

Proof. We prove this lemma by considering difference cases. First, assuming that there exists an element e such that $f^*(e) > \frac{1}{\alpha} f^*(OPT)$, XOS-RANDOM-SAMPLE has a probability of $\frac{1}{2}$ to return the most value agent. Hence, XOS-RANDOM-SAMPLE is 2α -approximate the optimal in this case.

Second, we consider the case that $f^*(e) \leq \frac{1}{\alpha} f^*(OPT)$ for all $e \in OPT$. The main idea is to show that there exists a $S' \subseteq S^*$ such that $c(S')$ is at most b and $f(S')$ is a good approximation to $f^*(OPT)$. Let us divide this case into two sub-cases.

- $c(S^*) > b$. In this case, since $c(e) \leq b$ for all $e \in E$, we can always find a subset $S' \subset S^*$ such that $\frac{b}{2} \leq c(S') \leq b$. By Lemma 6.3, we know $f(S') \geq t \cdot c(S') \geq \frac{v(\text{OPT}(T_1))}{\beta \cdot b} \cdot \frac{b}{2} \geq \frac{v(\text{OPT}(T_1))}{2\beta}$. As $f(\text{OPT}(S^*))$ is at least $f(S')$, we have $f(\text{OPT}(S^*)) \geq f(S') \geq \frac{v(\text{OPT}(T_1))}{2\beta} \geq \frac{\alpha-1}{8\alpha\beta} f^*(\text{OPT})$ with a probability of at least $\frac{1}{2}$.
- $c(S^*) \leq b$. Then $\text{OPT}(S^*) = S^*$. Let $S' = \text{OPT} \setminus T_1$, thus, $c(S') \leq c(\text{OPT}) \leq b$. By Lemma 6.2, we have $v(S') \geq \frac{\alpha-1}{4\alpha} f^*(\text{OPT})$ with a probability of at least $\frac{1}{2}$. Since $S^* \in \arg \max_{S \subseteq T_2} \{v(S) - t \cdot c(S)\}$, with a probability of at least $\frac{1}{2}$, we have

$$\begin{aligned}
f(\text{OPT}(S^*)) &= f(S^*) = v(S^*) \\
&\geq v(S^*) - t \cdot c(S^*) \\
&\geq v(S') - t \cdot c(S') \\
&\geq \frac{\alpha-1}{4\alpha} f^*(\text{OPT}) - \frac{v(\text{OPT}(T_1))}{\beta \cdot b} \cdot b \\
&\geq \frac{\alpha-1}{4\alpha} f^*(\text{OPT}) - \frac{f^*(\text{OPT})}{\beta} \\
&= \frac{\alpha \cdot \beta - \beta - 4\alpha}{4 \cdot \alpha \cdot \beta} f^*(\text{OPT})
\end{aligned}$$

In both cases, we run ADDITIVE-MECHANISM which has an approximation factor of 3 to $f(\text{OPT}(S^*))$. Therefore, the approximation ratio for the case that $f^*(e) \leq \frac{1}{\alpha} f^*(\text{OPT})$ for all $e \in \text{OPT}$ is

$$\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \min\left(\frac{\alpha-1}{8 \cdot \alpha \cdot \beta}, \frac{\alpha \cdot \beta - \beta - 4\alpha}{4 \cdot \alpha \cdot \beta}\right)$$

To combine with the first case, we conclude that the approximation of the mechanism is

$$\min\left(\frac{1}{2\alpha}, \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} \min\left(\frac{\alpha-1}{8 \cdot \alpha \cdot \beta}, \frac{\alpha \cdot \beta - \beta - 4\alpha}{4 \cdot \alpha \cdot \beta}\right)\right)$$

By setting $\alpha \approx 218$ and $\beta \approx 4.5$, we get the approximation of 436.

Lemma 6.5 (Lemma 3.1 in [3]). XOS-MECHANISM-MAIN is universally truthful.

References

1. Nima Anari, Geetika Goel, and Afshin Nikzad. Mechanism design for crowdsourcing: An optimal 1-1/e competitive budget-feasible mechanism for large markets. In *55th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 266–275. IEEE, 2014.
2. Lawrence M Ausubel. An efficient ascending-bid auction for multiple objects. *American Economic Review*, pages 1452–1475, 2004.
3. Xiaohui Bei, Ning Chen, Nick Gravin, and Pinyan Lu. Budget feasible mechanism design: from prior-free to bayesian. In *Proceedings of the forty-fourth Annual ACM Symposium on Theory of Computing (STOC)*, pages 449–458. ACM, 2012.
4. Sushil Bikhchandani, Sven de Vries, James Schummer, and Rakesh V Vohra. An ascending vickrey auction for selling bases of a matroid. *Operations research*, 59(2):400–413, 2011.
5. Hau Chan and Jing Chen. Truthful multi-unit procurements with budgets. In *the proceedings of the 10th International Conference on Web and Internet Economics (WINE)*, pages 89–105, 2014.
6. Ning Chen, Nick Gravin, and Pinyan Lu. On the approximability of budget feasible mechanisms. In *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 685–699. SIAM, 2011.
7. E.H. Clarke. Multipart pricing of public goods. *Public choice*, 11(1):17–33, 1971.

8. Gabrielle Demange, David Gale, and Marilda Sotomayor. Multi-item auctions. *The Journal of Political Economy*, pages 863–872, 1986.
9. Gagan Goel, Vahab Mirrokni, and Renato Paes Leme. Polyhedral clinching auctions and the adwords polytope. *Journal of the ACM (JACM)*, 62(3):18, 2015.
10. Gagan Goel, Afshin Nikzad, and Adish Singla. Allocating tasks to workers with matching constraints: truthful mechanisms for crowdsourcing markets. In *Proceedings of the companion publication of the 23rd international conference on World Wide Web companion*, pages 279–280, 2014.
11. T. Groves. Incentives in teams. *Econometrica: Journal of the Econometric Society*, pages 617–631, 1973.
12. Anna R Karlin and David Kempe. Beyond vcg: Frugality of truthful mechanisms. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 615–624. IEEE, 2005.
13. Robert Kleinberg and Seth Matthew Weinberg. Matroid prophet inequalities. In *Proceedings of the forty-fourth annual ACM symposium on Theory of computing*, pages 123–136. ACM, 2012.
14. Piotr Krysta and Jinshan Zhang. House markets with matroid and knapsack constraints. In *Proceedings of The 43rd International Colloquium on Automata, Languages and Programming (ICALP)*, 2016.
15. Alexander Schrijver. *Combinatorial Optimization. Algorithms and Combinatorics*, vol. 24. Springer, Berlin, 2003.
16. Yaron Singer. Budget feasible mechanisms. In *51st Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 765–774, 2010.
17. David NC Tse and Stephen V Hanly. Multiaccess fading channels. i. polymatroid structure, optimal resource allocation and throughput capacities. *IEEE Transactions on Information Theory*, 44(7):2796–2815, 1998.
18. W. Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of Finance*, 16(1):8–37, 1961.